Fiber graphs

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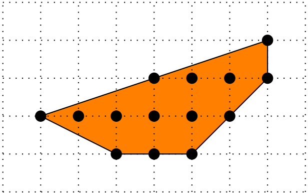
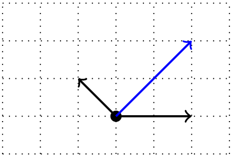
May 10, 2016

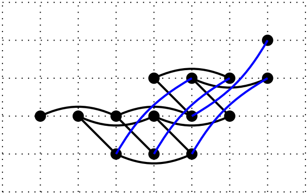
The keyplayer



finite set $\mathcal{F} \subseteq \mathbb{Z}^d$ with $\mathcal{F} = \mathrm{conv}_{\mathbb{Q}}(\mathcal{F}) \cap \mathbb{Z}^d$



moves $\mathcal{M} \subseteq \mathbb{Z}^d$ without multiples



fiber graph $\mathcal{F}(\mathcal{M})$

Fiber graph

The *fiber graph* $\mathcal{F}(\mathcal{M})$ is the graph on $\mathcal{F}$ where two nodes $u, v \in \mathcal{F}$ are adjacent if $u - v \in \pm\mathcal{M}$.

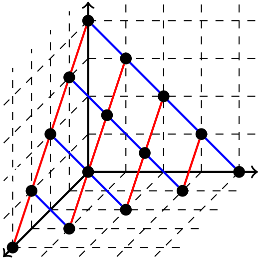
In many applications, $\mathcal{F}$ is given implicit as *fiber*

$$\mathcal{F} = \mathcal{F}_{A,b} := \{u \in \mathbb{N}^d : Au = b\}.$$

Example

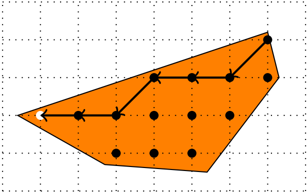
$A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 4 \end{bmatrix}$.

$$\mathcal{F}_{A,b} = \left\{ \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \right\}$$



$$\mathcal{M} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\} \cup \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$$

Fiber graphs in discrete optimization



$$\max\{f(u) : Au = b, u \in \mathbb{N}^d\}$$
$$= \max\{f(u) : u \in \mathcal{F}_{A,b}\}$$

Optimality certificates

A finite set $\mathcal{M} \subseteq \ker_{\mathbb{Z}}(A)$ is an *optimality certificate* if there is for every non-optimal u_0 a move $m \in \mathcal{M}$ such that $u_0 + m \in \mathcal{F}_{A,b}$ and

$$f(u_0 + m) > f(u_0)$$

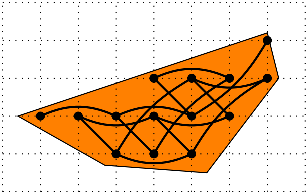
Graver bases

- Optimality certificate for separable convex functions
- Polynomial-time algorithm (in binary encoding length of Graver basis)
- Conjecture (Hemmecke, W.): Edge-Connectivity best-possible

Fiber graphs in statistics

- $A \in \mathbb{Z}^{m\times d}$, $b \in \mathbb{Z}^m$
- $\pi : \mathcal{F}_{A,b} \to [0,1]$ mass function
- $f : \mathcal{F}_{A,b} \to \mathbb{R}^m$ any function

$$\mathbb{E}_\pi(f) = \sum_{u\in\mathcal{F}_{A,b}} \pi(u)\cdot f(u)$$



Markov chain Monte Carlo

- Construct a finite set $\mathcal{M} \subseteq \mathbb{Z}^d$ such that $\mathcal{F}_{A,b}(\mathcal{M})$ is connected
- Construct a random walk W on $\mathcal{F}_{A,b}(\mathcal{M})$ that converges to π
- Collect samples $u_1,\dots,u_n \in \mathcal{F}_{A,b}$ with W and compute $\frac{1}{n}\sum_{i=1}^n f(u_i)$
- Ergodic theorem: $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n f(u_i) = \mathbb{E}_\pi(f)$ (almost sure)

Classic MCMC issue: How large should n be?

Connectedness

$\mathcal{M}$ Markov basis if $\mathcal{F}(\mathcal{M})$ connected.

Markov vs lattice basis

A Markov basis has far more elements than just a lattice basis of $\ker_{\mathbb{Z}}(A)$!

Diaconis, Sturmfels; 1998

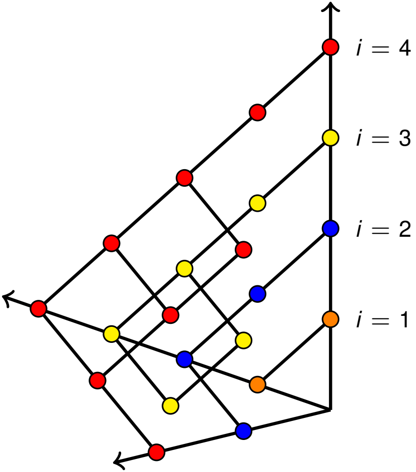
Let $A \in \mathbb{Z}^{m \times d}$ with $\ker_{\mathbb{Z}}(A) \cap \mathbb{N}^d = \{0\}$. A set $\mathcal{M} \subseteq \ker_{\mathbb{Z}}(A)$ is a Markov basis for $\mathcal{F}_{A,b}$ for all $b \in \mathbb{Z}^m$ if and only if $\mathcal{I}_{\mathcal{M}} = \mathcal{I}_A$ in $\mathbb{k}[x_1, \ldots, x_d]$.

Computation of Markov bases

- Fundamental problem of algebraic statistics!
- Generating sets of toric ideals can be computed (Gröbner methods)
- But: $\mathbb{NP}$-hard in general
- State-of-the-art: *Project-and-lift* algorithm by Hemmecke/Malkin

Dilation

Let's say we have Markov basis $\mathcal{M}$ for $\mathcal{F}_{A,b}$ for all $b \in \mathbb{Z}^m$
$\rightarrow$ Study sequence $(\mathcal{F}_{A,i \cdot b}(\mathcal{M}))_{i \in \mathbb{N}}$



Theorem (W.; 2015)

Let $\mathcal{M}$ be a Markov basis of A and $b \in \mathbb{Z}^m$, then

$$h(\mathcal{F}_{A,i \cdot b}(\mathcal{M})) \leq \frac{C}{i}.$$

Theorem (Stanley, W.; 2016)

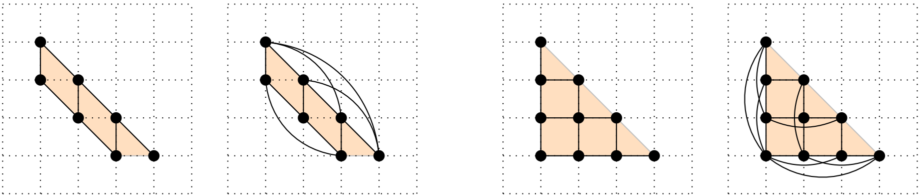
Let $\mathcal{M}$ be a Markov basis of A and $b \in \mathbb{Z}^m$, then

$$C' \cdot i \leq \mathrm{diam}(\mathcal{F}_{A,i \cdot b}(\mathcal{M})) \leq C \cdot i.$$

Speed things up

Compressed fiber graphs

The *compressed fiber graph* $\mathcal{F}^c(\mathcal{M})$ is the graph on $\mathcal{F}$ where two nodes $u, v \in \mathcal{F}$ are adjacent if there is $m \in \mathcal{M}$ and $\lambda \in \mathbb{Z}$ such that $u - v = \lambda m$.



Theorem (Stanley, W.; 2016)

$\mathrm{diam}(\mathcal{F}^c_{A,b}(\mathcal{M})) \leq C$ *for all* $b \in \mathbb{Z}^m$.

Heat-bath random walk on compressed fiber graphs mixes rapidly in many situations.

$\{$Fiber graphs$\} \subsetneq \{$Simple graphs$\}$?

Observation

Every simple graph is a fiber graph!

Let $G = (\{v_1, \dots, v_n\}, E)$ be a graph and define $\mathcal{F} := \{e_1, \dots, e_n\} \subseteq \mathbb{Z}^n$ and $\mathcal{M} := \{e_i - e_j : v_i v_j \in E\}$, then $G \cong \mathcal{F}(\mathcal{M})$.

Fiber dimension

The *fiber dimension* $\mathrm{fdim}(G)$ of G is the smallest $d \in \mathbb{N}$ such that there exists a normal set $\mathcal{F} \subseteq \mathbb{Z}^d$ and a set of moves $\mathcal{M} \subseteq \mathbb{Z}^d$ with $G \cong \mathcal{F}(\mathcal{M})$.

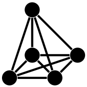
Embedding $\Rightarrow \mathrm{fdim}(G) \leq |V(G)| - 1$

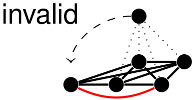
Euclidean Dimension; Erdős, Simonovits; 1980

The *euclidean dimension* $\mathrm{edim}(G)$ of G is the smallest $d \in \mathbb{N}$ such that G is isomorphic to a unit distance graph in $\mathbb{R}^d$.

$\mathrm{edim}(K_5) = 5$

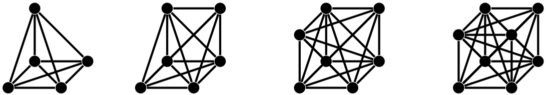
$\mathrm{fdim}(K_5) \leq 3$



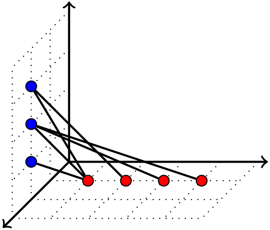


Fiber dimension

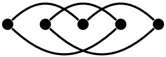
- $\mathrm{fdim}(K_n) = \lceil \log_2(n) \rceil$



- $\mathrm{fdim}(G) \leq 2 \cdot \chi(G) - 1$



- $\mathrm{fdim}(C_n) = \begin{cases} 1, \text{ if } n \notin \{3,4,6\} \\ 2, \text{ if } n \in \{3,4,6\} \end{cases}$

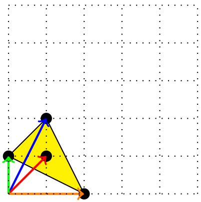


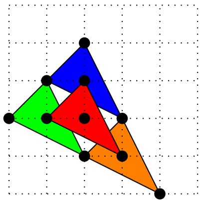
Proof sketch: $n \notin \{3,4,6\} \Rightarrow \phi(n) > 2 \Rightarrow \exists 1 < k < \frac{n}{2}$ with $\gcd(k,n) = 1$
$\Rightarrow C_n \cong [n](\{k, n-k\})$

Distinct pair-sum polytopes

Distinct pair-sum polytope

A lattice polytope $\mathcal{P} \subseteq \mathbb{Q}^d$ with $n := |\mathcal{P} \cap \mathbb{Z}^d|$ is a *distinct pair-sum polytope* if $|\mathcal{P} \cap \mathbb{Z}^d + \mathcal{P} \cap \mathbb{Z}^d| = \binom{n}{2} + n$.





Theorem (Choi, Lam, Reznick; 2002)

Let $\mathcal{P} \subseteq \mathbb{Q}^d$ be a distinct pair-sum polytope, then $|\mathcal{P} \cap \mathbb{Z}^d| \leq 2^d$.

New Proof: Let $\mathcal{F} := \mathcal{P} \cap \mathbb{Z}^d$, $n := |\mathcal{F}|$, and $\mathcal{M} = \{u - v : u, v \in \mathcal{F}\}$. Since $\mathcal{P}$ is distinct pair-sum polytope, $\mathcal{M}$ is a set of moves and $K_n \cong \mathcal{F}(\mathcal{M})$. Thus $\mathrm{fdim}(K_n) \leq d$.

Open problems

Fiber graphs

- Minimal degree?
- Edge-connectivity?

Fiber dimension

- Are there graphs with $\text{fdim}(G) = |V(G)| - 1$?
- Fiber dimension of trees? When is $\text{fdim}(T) = 3$?
- How to compute $\text{fdim}(G)$? Is this even Turing-computable?
- Complexity?

Thanks!